PHYSICS 1

Simple Harmonic Motion
Simple Harmonic Motion

What I Absolutely Have to Know to Survive the AP* Exam

- Whenever the acceleration of an object is proportional to its displacement and is oppositely directed, the object will move with Simple Harmonic Motion (SHM). In harmonic motion there is always a restorative force, which acts in the opposite direction of the displacement. The restorative force changes during oscillation and depends on the position of the object. In a spring the force is given by Hooke’s Law, in a pendulum it is the component of gravity along the path, or directly opposite that of the displacement.
- Remember that a spring constant tells you how rigid the spring is and how much force per unit distance is needed to elongate or compress the spring.
- The oscillating object does not lose any energy in SHM. Friction is assumed to be negligible.
- Objects in simple harmonic motion do not obey kinematic equations of motion because the acceleration is not constant. As a spring compresses, the force (and hence acceleration) increases. As a pendulum swings, the tangential component of the force of gravity changes, so the acceleration changes.
- The period, $T$, is the amount of time it takes for the harmonic motion to repeat itself, or for the object to complete one full cycle. In SHM, $T$ is the time it takes the object to return to its exact starting point and starting direction.
- The frequency, $f$, is the number of cycles an object goes through in 1 second. Frequency is measured in Hertz (Hz). 1 Hz = 1 cycle per sec.
- The amplitude, $A$, is the distance from the equilibrium (or center) point of motion to either its lowest or highest point (end points). The amplitude, therefore, is half of the total distance covered by the oscillating object. The amplitude can vary in harmonic motion but is constant in SHM.
- The kinetic energy and the speed are at a maximum at the equilibrium point, but the potential energy and restorative force is zero there.
- At the end points the potential energy is at a maximum, while the kinetic energy and speed are zero. At the end points the restorative force and acceleration are at a maximum.
- In SHM since energy is conserved the best method for calculating position and velocity is to set the total energy equal to the sum of kinetic and potential energies. Similarly force and acceleration are best calculated by using Newton’s second law $\ddot{a} = \frac{\Sigma F}{m}$. 

<table>
<thead>
<tr>
<th>Key Formulas and Relationships</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hooke’s Law [ F = kx ]</td>
</tr>
</tbody>
</table>

where
- $x =$ displacement of object (m)
- $k =$ force or spring constant (N/m)

$x = A \cos(2\pi ft)$

where
Simple Harmonic Motion

\( x = \) displacement of object (m)  
\( A = \) amplitude = maximum displacement for equilibrium displacement (m)  
\( f = \) frequency (hertz, Hz or oscillations/sec)  
\( t = \) time (s)

Other equations

\[
T = \frac{2\pi}{\omega} = \frac{1}{f} \\
T = \frac{2\pi}{\frac{1}{T} = \frac{\omega}{2\pi}}
\]

where

\( T = \) period of oscillation (seconds)  
\( f = \) frequency (hertz, Hz or oscillations/sec)

SHM of an object vibrating on a spring

\[
T_s = 2\pi \sqrt{\frac{m}{k}}
\]

where

\( k = \) spring constant  
\( m = \) mass of object of spring (kg)

Energy in SHM

For an object on a spring

\[
U_s = \frac{1}{2} kx^2 \\
K = \frac{1}{2} mv^2
\]

where

\( U = \) elastic potential energy (J)  
\( K = \) kinetic energy (J)

\[
E_{total} = \frac{1}{2} kA^2
\]

Total energy in SHM is proportional to \( A^2 \).

SHM for a simple pendulum

\[
T_p = 2\pi \sqrt{\frac{L}{g}}
\]

where
Simple Harmonic Motion

pendulum is swinging at small angles (θ < 15°)
l = length of simple pendulum (meters)
g = acceleration due to gravity = 9.8 m/s²

Example 1

A spring of negligible mass and of spring constant 245 N/m is hung vertically and not extended. A mass of 2.5 kg is attached to the spring and it stretches a distance $x_o$.

(a) What is $x_o$ in meters?

Use Hooke’s law

$F = kx$

$x_o = \frac{mg}{k} = \frac{(2.5 \text{ kg})(9.8 \text{ m/s}^2)}{245 \text{ N/m}}$

$x_o = 0.10 \text{ m}$

Now the spring is pulled down an additional distance $x = 0.06$ meters.

(b) What is the frequency of the motion when the spring is released from $(x + x_o)$?

$f' = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$

$f' = \frac{1}{2\pi} \sqrt{\frac{245 \text{ N/m}}{2.5 \text{ kg}}}$

$f' = 1.58 \text{ Hz}$

Note to presenter: Student handouts contain multiple choice and free response questions, but they do not contain the TIPER’s. Please use the TIPER’s (found at the end of your handout) as you see fit in order to provide the students a rich understanding of the material. A document camera will enable you to share information with students.
Questions 1 – 2
An object of mass 0.8 kg is initially at rest on a horizontal frictionless surface. The object is acted upon by a horizontal force $F$, the magnitude of which varies as function of the displacement of the object $d$ as shown in the graph.

1. Which of the following would be most likely to produce the force shown in the graph?

a) a string tied to a falling mass
b) a stretched spring
c) a magnet placed in the direction of the displacement
d) kinetic friction

The force is decreasing as the object is displaced

The graph can only be produced by a variable force that gets weaker as the object moves – a stretched spring that pulls the object as it relaxes is the only one capable of producing such a graph

2. The amount of work done by the force $F$ in displacing the object 0.2 m is most nearly

a) 0 J
b) 1 J
c) 2 J
d) 4 J

Work done by a variable force is equal to the area under the curve of $F/d$ graph

The graph forms a triangle $A = \frac{1}{2}bh$ so the work is

$$W = \frac{1}{2} \cdot (0.2m) \cdot (20N) = 2J$$
Simple Harmonic Motion

Questions 3 – 4

A simple pendulum is constructed from a string of length $l$ and a bob of mass $m$ as shown in the diagram. It is released from rest at point I, which is a vertical distance $y$ from the equilibrium position. The bob has zero potential energy at point II. Friction and drag are negligible.

3. The speed of the pendulum bob when it is in position II is most nearly

a) $\frac{1}{2} ml^2$

b) $\sqrt{2gy}$

c) $\sqrt{2g(l-y)}$

d) $\sqrt{2g(l^2-y^2)}$

Conservation of Energy

At I, the bob has zero kinetic energy and at II the bob has zero potential energy. Potential energy at I is converted to kinetic energy at II (energy is conserved).

$U_{e,i} = K_f$

$mgy = \frac{1}{2}mv^2$

$v = \sqrt{2gy}$
Simple Harmonic Motion

4. Which of the following graphs best shows the total energy of the pendulum bob as a function of displacement?

Conservation of Energy

Since there are no non-conservative forces acting on it, the total energy of the bob remains constant. Graph E shows this.

Graph E: E

5. A block is attached to a horizontal spring and put into simple harmonic motion with amplitude of 6.0 cm. When the block is 3.0 cm from the equilibrium position, what is the ratio of total mechanical energy to kinetic energy?

A. \(\frac{1}{4}\)
B. \(\frac{1}{3}\)
C. \(\frac{1}{2}\)
D. \(\frac{3}{4}\)

Energy in SHM

Ratio of total mechanical energy to kinetic energy: D
Simple Harmonic Motion

\[ U = \frac{1}{2} kx^2 = \frac{1}{2} k(3)^2 \approx 9 \]
\[ E_{\text{total}} = \frac{1}{2} kA^2 = \frac{1}{2} k(6)^2 \approx 36 \]
\[ E_{\text{total}} = U + K \]
\[ K = E_{\text{total}} - U \approx 27 \]
\[ K = \frac{27}{3} = 9 \]
\[ E_{\text{total}} = \frac{36}{4} = 9 \]

6. What is the total energy of a 1.0 kg object on a horizontal spring with amplitude of 0.10 m and frequency of 5 Hz?

A. \( \frac{1}{10} \pi^2 \)
B. \( \frac{1}{2} \pi^2 \)
C. \( \pi^2 \)
D. \( 5\pi^2 \)

Energy in SHM and angular frequency

\[ f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \]
\[ k = (2\pi f)^2 m \]
\[ k = (2\pi 5)^2 (1 \text{ kg}) = 100\pi^2 \]
\[ E_{\text{total}} = \frac{1}{2} kA^2 = \frac{1}{2} (100\pi^2)(0.1)^2 = \frac{1}{2} \pi^2 \]
Simple Harmonic Motion

7. A mass-spring system is set up so that it exhibits SHM with an amplitude of 6.0 cm. How are the total mechanical energy and the system’s maximum displacement affected if the amplitude was doubled to 12.0 cm?

<table>
<thead>
<tr>
<th>Total Mechanical Energy</th>
<th>Maximum Displacement</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. 4 times larger</td>
<td>4 times larger</td>
</tr>
<tr>
<td>B. 4 times larger</td>
<td>2 times larger</td>
</tr>
<tr>
<td>C. 2 times larger</td>
<td>2 times larger</td>
</tr>
<tr>
<td>D. 4 times larger</td>
<td>not changed</td>
</tr>
</tbody>
</table>

Energy in SHM and Equations of SHM

\[ E_{\text{total}} = \frac{1}{2} kA^2 \]

Doubling the amplitude will increase energy by \(2^2 = 4\) times greater.

\[ x = A \cos(2\pi ft) \]

Doubling the amplitude will double the displacement.

8. Mass-spring system A has a frequency \(f_A\). Mass spring system B has a frequency \(f_B = 2f_A\). Both springs have the same length and the same spring constant. How are \(m_A\) and \(m_B\) related?

A. \(m_A = \frac{1}{4} m_B\)
B. \(m_A = \frac{1}{2} m_B\)
C. \(m_A = \sqrt{2} m_B\)
D. \(m_A = 4m_B\)
### Simple Harmonic Motion

<table>
<thead>
<tr>
<th>SHM of an object vibrating on a spring</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f = {\frac{1}{2\pi}}\sqrt{\frac{k}{m}}$</td>
</tr>
<tr>
<td>$f_B = 2f_A$</td>
</tr>
<tr>
<td>$\frac{1}{2\pi}\sqrt{\frac{k}{m_B}} = 2\left(\frac{1}{2\pi}\sqrt{\frac{k}{m_A}}\right)$</td>
</tr>
<tr>
<td>$\frac{1}{m_B} = 4\left(\frac{1}{m_A}\right)$</td>
</tr>
<tr>
<td>$m_A = 4m_B$</td>
</tr>
</tbody>
</table>

9. The pendulum in a clock increases in length slightly as the clock’s temperature increases. How will this affect the clock’s time keeping abilities?

A. The pendulum swings slower, so the clock runs slower.
B. The clock’s molecules vibrate faster, so the clock runs faster.
C. The pendulum swings faster, so the clock runs faster.
D. The longer pendulum can keep more accurate time.

<table>
<thead>
<tr>
<th>SHM for a simple pendulum</th>
<th>The increase in temperature, thus the increase in length, will cause the period to increase. The clock will run slower than normal.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T = 2\pi\sqrt{\frac{l}{g}}$</td>
<td>A</td>
</tr>
</tbody>
</table>

10. A simple pendulum on the Earth’s surface, a distance of $R$ from the center of the Earth, has a period of 1.0 second. The pendulum is moved to a distance $2R$ from the Earth’s center. What is the period at the new location?

A. 1.0s  
B. 2.0s  
C. 3.0s  
D. 4.0s

| SHM for a simple pendulum and Universal Gravitation | Since the distance is doubled, the acceleration due to gravity is $\frac{1}{4}$ of its original value | b |
Simple Harmonic Motion

\[ T = 2\pi \sqrt{\frac{l}{g}} \]

\[ T' = 2\pi \sqrt{\frac{l}{g^2/4}} = 2 \left( 2\pi \sqrt{\frac{l}{g}} \right) = 2T \]

\[ T' = 2(1\text{sec}) = 2\text{sec} \]

11. A block of mass 3.0 kg is hung from a spring, causing it to stretch 12 cm at equilibrium, as shown above. The 3.0 kg block is then replaced by a 4.0 kg block, and the new block is released from the position shown above, at which the spring is unstretched. How far will the 4.0 kg block fall before its direction is reversed?

A. 9 cm  
B. 18 cm  
C. 24 cm  
D. 32 cm

Calculate the spring constant first:

\[ F = kx \implies k = \frac{F}{x} = \frac{mg}{x} = \frac{(3\text{kg})(10 \frac{m}{s^2})}{0.12 \text{m}} = 250 \frac{N}{m} \]

Energy is conserved, so calculate the maximum potential energy (occurs at maximum elongation where \(KE\) is equal to zero as the spring must stop to change direction):

\[ U = mgh = (4\text{kg}) \left(10 \frac{m}{s^2}\right)h = \frac{1}{2}kx^2 \]

, so the new expression becomes
Simple Harmonic Motion

\[
(4 \text{ kg}) \left( 10 \frac{\text{m}}{\text{s}^2} \right) h = \frac{1}{2} \left( 250 \frac{\text{N}}{\text{m}} \right) h^2
\]

\[
40 = 125h \quad \therefore h = 0.32
\]

12.

A sphere of mass \( m_1 \), which is attached to a spring, is displaced downward from its equilibrium position as shown above left and released from rest. A sphere of mass \( m_2 \), which is suspended from a string of length \( \ell \), is displaced to the right as shown above right and released from rest so that it swings as a simple pendulum with small amplitude. Assume that both spheres undergo simple harmonic motion. Which of the following is true for both spheres?

A. The maximum kinetic energy is attained as the sphere passes through its equilibrium position.
B. The maximum kinetic energy is attained as the sphere reaches its point of release.
C. The minimum gravitational potential energy is attained as the sphere passes through its equilibrium position.
D. The maximum gravitational potential energy is attained when the sphere reaches its point of release.

Page 4

A

SHM for a simple pendulum and spring

Answer (B) is clearly wrong, because at the point of release, each sphere will be momentarily at rest. Answer (C) sounds pretty good; in fact, it is correct for the pendulum’s mass; but for the spring’s mass, the minimum gravitational energy is attained at its lowest point, not the midpoint. For the same reason answer (D) is wrong; it is true for the pendulum’s mass, but not for the spring’s.
Finally, the total energy doesn’t have either a maximum or a minimum; it is constant. Both masses have the same energy throughout (in the idealized case that air resistance is zero.)

Questions 13–14
A block oscillates without friction on the end of a spring as shown. The minimum and maximum lengths of the spring as it oscillates are, respectively, $x_{\text{min}}$ and $x_{\text{max}}$ located at positions I and III.

13. The graphs below can represent quantities associated with the oscillation as functions of the length $x$ of the spring. Which of the following graphs best shows the total mechanical energy of the block-spring system as a function of displacement?
Simple Harmonic Motion

Conservation of Energy

<table>
<thead>
<tr>
<th>Conservation of Energy</th>
<th>Since there are no non-conservative forces acting on it, the total energy of the block remains constant. Graph E shows this.</th>
<th>E</th>
</tr>
</thead>
</table>

14. Which of the following graphs best shows the kinetic energy of the block as a function of displacement?

\[ E = KE + PE = KE + \frac{1}{2} kx^2 \]

Total energy = constant, so \( KE = (\text{constant}) - \frac{1}{2} kx^2 \), which is an upside-down parabola.

Alternatively, use the fact that the speed at the endpoints is zero, so the \( KE \) graph must hit the \( x \)-axis at the endpoints. The answer cannot be B) because the force varies smoothly, and so the acceleration varies smoothly. In A), we have the acceleration jumping suddenly at the equilibrium point \( (x = 0) \); this is impossible.

Conservation of Energy

<table>
<thead>
<tr>
<th>Conservation of Energy</th>
<th>[ E = KE + PE = KE + \frac{1}{2} kx^2 ] Total energy = constant, so ( KE = (\text{constant}) - \frac{1}{2} kx^2 ), which is an upside-down parabola. Alternatively, use the fact that the speed at the endpoints is zero, so the ( KE ) graph must hit the ( x )-axis at the endpoints. The answer cannot be B) because the force varies smoothly, and so the acceleration varies smoothly. In A), we have the acceleration jumping suddenly at the equilibrium point ( (x = 0) ); this is impossible.</th>
<th>D</th>
</tr>
</thead>
</table>

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Simple Harmonic Motion

Question 1 (12 pts)
1. A block of mass $m$ is attached to an ideal spring of spring constant $k$, the other end of which is fixed. The block is on a level, frictionless surface as shown in the diagram. At time $t_0$, the block is set into simple harmonic motion of period $T$ by an external force pushing it to the right, giving the block initial velocity $v_0$. Express all answers in terms of the given quantities and fundamental constants.

A. Determine the amplitude of the block’s motion.

<table>
<thead>
<tr>
<th>(4 points max)</th>
<th>1 point For applying conservation of energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{\text{initial}} = U_{s,\text{max}}$</td>
<td>1 point For equating the initial kinetic energy of the block with the maximum elastic potential energy</td>
</tr>
<tr>
<td>$\frac{1}{2}mv_0^2 = \frac{1}{2}kA^2$</td>
<td>1 point For correctly substituting the definitions of kinetic and elastic potential energy</td>
</tr>
<tr>
<td>$A = \sqrt{\frac{mv_0^2}{k}}$</td>
<td>1 point For the correct answer in terms of the given quantities and fundamental constants</td>
</tr>
</tbody>
</table>

B. On the graph below, plot the kinetic energy of the block-spring-Earth system as a function of time for two periods. Explicitly label any intercepts, asymptotes, maxima, or minima with values as appropriate.
Simple Harmonic Motion

(5 points max)

sinusoidal graph, positive, with max KE at 0, T/2, T, 3T/2, and 2T

1 point for a reasonable sinusoidal graph

1 point for a graph that shows positive kinetic energy (graph is always positive)

1 point for showing the maximum kinetic energy and the minimum kinetic energy is zero

1 point for correctly showing that the maximum kinetic energy occurs at t = 0, T/2, T, 3T/2, and 2T

1 point for correctly showing that the minimum kinetic energy occurs at t = T/4, 3T/4, 5T/4, and 7T/4.

C. The block is stopped and a second identical block is glued on top of the first. The blocks are returned to simple harmonic motion with the same initial velocity as before.

i. How does the amplitude of the motion of the two blocks together compare to the amplitude found in part A?
Simple Harmonic Motion

Greater than __________  Less than ________  Equal ________

ii. Justify your answer.

(2 points max)

Greater than __x__

the increased mass increases the kinetic energy when the block is set in motion, meaning the spring will be compressed and stretch more using the equation from part A, as the mass increases so does the amplitude

\[ x_{\text{max}} = \sqrt{\frac{mv_0^2}{k}} \]

1 point For correctly indicating the amplitude increases when the mass increases

1 point for any reasonable justification

D. The blocks are stopped and the ideal spring is replaced by a nonlinear spring. Will the blocks undergo simple harmonic motion? Why or why not?

(1 point max)

No, the motion will not be simple harmonic motion as the nonlinear spring does not obey Hooke’s Law, as the force so the stretch. There is no longer a linear relationship between force and stretch.

1 point For stating that the nonlinear spring will not obey Hooke’s law
Simple Harmonic Motion

Question 2 (7 pts)

A spring that can be assumed to be ideal hangs from a stand, as shown above. The spring constant, $k$, of an ideal spring is defined as the force per unit length and differs from one spring to another. It can be measured in both a static (motionless) and dynamic (in motion) mode.

A. You wish to determine experimentally the spring constant $k$ of the spring in a static (motionless) situation.

i. What additional, commonly available equipment would you need?

<table>
<thead>
<tr>
<th>(2 points max)</th>
<th>1 point For indicating a device to measure distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Meter stick or device to measure distance</td>
<td>Calibrated mass or a Force Probe</td>
</tr>
<tr>
<td>1 point For indicating a known mass or force probe</td>
<td></td>
</tr>
</tbody>
</table>

ii. What measurements would you make?

<table>
<thead>
<tr>
<th>(2 points max)</th>
<th>1 point For measuring the force</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hang the known mass on the bottom of the spring and measure the extension of the spring</td>
<td>Or</td>
</tr>
<tr>
<td>Pull on the spring with a force probe and measure the extension of the spring</td>
<td></td>
</tr>
<tr>
<td>1 point For measuring the distance stretched</td>
<td></td>
</tr>
</tbody>
</table>

iii. How would $k$ be determined from these measurements?

<table>
<thead>
<tr>
<th>(1 point max)</th>
<th>1 point For using Hooke’s Law and solving for $k</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F = -kx$</td>
<td>$mg = k\Delta x$</td>
</tr>
<tr>
<td>$k = \frac{mg}{\Delta x}$</td>
<td></td>
</tr>
</tbody>
</table>
Simple Harmonic Motion

B. You wish to determine experimentally the spring constant \( k \) of the spring in a dynamic (moving) situation.

iv. What additional, commonly available equipment would you need?

<table>
<thead>
<tr>
<th>(2 points max)</th>
<th>1 point For indicating a stopwatch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stopwatch</td>
<td></td>
</tr>
<tr>
<td>Calibrated mass</td>
<td>1 point For indicating a known mass</td>
</tr>
</tbody>
</table>

v. What measurements would you make?

<table>
<thead>
<tr>
<th>(2 points max)</th>
<th>1 point For setting the known mass in SHM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hang the known mass on the bottom of the spring, set it into simple harmonic motion and measure the period</td>
<td>1 point For measuring the period</td>
</tr>
</tbody>
</table>

vi. How would \( k \) be determined from these measurements?

<table>
<thead>
<tr>
<th>(1 points max)</th>
<th>1 point For using the expression for the period of a spring and solving for ( k )</th>
</tr>
</thead>
</table>
| \[
\frac{T}{2\pi} = \sqrt{\frac{m}{k}} \\
\frac{m}{k} = \frac{T^2}{4\pi^2} \\
k = \frac{4\pi^2m}{T^2}
\] |                                    |

C. Assume that the spring constant is determined to be 500 N/m. A 2.0-kg mass is attached to the lower end of the spring and released from rest. Determine the frequency of oscillation of the mass.

<table>
<thead>
<tr>
<th>(2 points max)</th>
<th>1 point For indicating the frequency is the inverse of the period</th>
</tr>
</thead>
</table>
| \[
\frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \left( \sqrt{\frac{500 \text{ N/m}}{2.0 \text{ kg}}} \right) = 2.5 \text{ Hz}
\] | 1 point For correct answer with units and reasonable number of significant digits |

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In an experiment, students are to calculate the spring constant $k$ of a vertical spring in a small jumping toy that initially rests on a table. When the spring in the toy is compressed a distance $x$ from its uncompressed length $L_0$ and the toy is released, the top of the toy rises to a maximum height $h$ above the point of maximum compression. The students repeat the experiment several times, measuring $h$ with objects of various masses taped to the top of the toy so that the combined mass of the toy and added objects is $m$. The bottom of the toy and the spring each have negligible mass compared to the top of the toy and the objects taped to it.

(a) Derive an expression for the height $h$ in terms of $m$, $x$, $k$, and fundamental constants.

With the spring compressed a distance $x = 0.020$ m in each trial, the students obtained the following data for different values of $m$.

<table>
<thead>
<tr>
<th>$m$ (kg)</th>
<th>$h$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.020</td>
<td>0.49</td>
</tr>
<tr>
<td>0.030</td>
<td>0.34</td>
</tr>
<tr>
<td>0.040</td>
<td>0.28</td>
</tr>
<tr>
<td>0.050</td>
<td>0.19</td>
</tr>
<tr>
<td>0.060</td>
<td>0.18</td>
</tr>
</tbody>
</table>

(b)

i. What quantities should be graphed so that the slope of a best-fit straight line through the data points can be used to calculate the spring constant $k$?
ii. Fill in one or both of the blank columns in the table with calculated values of your quantities, including units.

(c) On the axes below, plot your data and draw a best-fit straight line. Label the axes and indicate the scale.

(d) Using your best-fit line, calculate the numerical value of the spring constant.

(e) Describe a procedure for measuring the height $h$ in the experiment, given that the toy is only momentarily at that maximum height.
Question 1

15 points total

(a) 3 points

For a correct statement of conservation of mechanical energy (which might be implied in the next statement) 1 point

For correct energy expressions set equal 1 point

\[ mgh = \frac{1}{2} kx^2 \]

For solving for \( h \), consistent with the energy equation 1 point

\[ h = \frac{kx^2}{2mg} \]

(b)

(i) 2 points

For a correct combination of \( 1/m \) and \( h \) or \( 1/h \) and \( m \), with or without constants 2 points

Notes:
- If both \( 1/m \) and \( 1/h \) were chosen, only 1 point was earned.
- If part (b)(i) contained quantities that would not yield a correct graph for part (c), no credit was given for parts (b)(ii) through (d). But if part (b)(i) was left blank, parts (b)(ii) through (d) were examined for correct results.

The example of \( h \) versus \( 1/m \) is used in the remainder of the scoring guideline.

(ii) 2 points

\[
\begin{array}{|c|c|c|}
\hline
1/m (kg^{-1}) & m (kg) & h (m) \\
\hline
50 & 0.020 & 0.49 \\
33 & 0.030 & 0.34 \\
25 & 0.040 & 0.28 \\
20 & 0.050 & 0.19 \\
17 & 0.060 & 0.18 \\
\hline
\end{array}
\]

For correctly filling in the table with the appropriate data 1 point

For including the correct units in the table 1 point
(c) 4 points

For correctly plotting appropriate data (all five data points plotted correctly, assuming the data represented an inverse relationship between m and h) 1 point
For correctly drawing a best-fit straight line (a single straight line with data points reasonably scattered above and below the line) 1 point
For correctly labeling both axes 1 point
For correctly indicating the scale on both axes 1 point

(d) 2 points

For correctly calculating the slope from points on the line 1 point
For example: slope = \( \frac{(0.42 - 0.10) \text{ m}}{(40 - 10) \text{ kg}^{-1}} \) = \( \frac{0.32 \text{ m}}{30 \text{ kg}^{-1}} \) = \( 1.07 \times 10^{-2} \text{ m} \cdot \text{kg} \)
For a correct numerical value of the spring constant 1 point
From (a), \( h = \frac{kx^2}{2mg} \), so the slope of the line = \( \frac{kx^2}{2g} \)

\[
k = \frac{2g(\text{slope})}{x^2}
\]

\[
k = \frac{2\left(9.8 \text{ m/s}^2\right)\left(1.07 \times 10^{-2} \text{ m} \cdot \text{kg}\right)}{(0.02 \text{ m})^2}
\]

\[
k = 524 \text{ N/m} \quad (535 \text{ N/m using } g = 10 \text{ m/s}^2)
\]
Note: Values between 450 N/m and 550 N/m were accepted.
(e) 2 points

For a correct, complete procedure

Examples: Use a meter stick and a visual recorder, such as the eye, a video camera, or a camera.
Use a sonic range finder and appropriate computer software.
Measure the time (either to rise or for a round trip) and give the correct appropriate kinematics equations (the equations must be included in the answer).

One point was awarded for a partially correct or incomplete procedure.

Examples: Use a meter stick.
Use a photogate.
The springs shown below have the same unstretched length and the same spring constant \( k \). All of the carts are identical and are on horizontal frictionless surfaces. In case C, there are two springs attached end-to-end. The carts are given a small horizontal displacement and then released.

Rank these cases on the basis of the oscillation frequency of the carts.

Greatest 1 _______ 2 _______ 3 _______ 4 _______ Least

OR, The frequencies are the same for all these cases.  ___

OR, We cannot determine the ranking for the frequencies of these systems.  ___

Please explain your reasoning.

Answer: \( A = B > D > C \)

The frequency of the oscillations will be greater if the restoring force is greater for a given displacement. For each spring attached to the cart the restoring force for a displacement \( \Delta x \) is equal to the spring constant of that spring times the displacement, and does not depend on whether the spring is stretched or compressed. So for cases A and B, the restoring force is \( (k + k)\Delta x \), and for case D it is \( k\Delta x \). For case C, when the cart is displaced a distance \( \Delta x \) each spring is only displaced by \( 1/2\Delta x \), so the restoring force is only \( 1/2k\Delta x \).
A frictionless cart of mass $m$ is attached to a spring with spring constant $k$. When the cart is displaced from its rest position and released, it oscillates with a period $\tau$ that is given by

$$\tau = 2\pi \sqrt{\frac{m}{k}}.$$

The graph of the position of the cart as a function of time is shown below for Experiment A. Graphs for two other experiments are shown below this. The same spring is used in all three experiments.

(1) Compared to Experiment A, in Experiment B the cart has

a) twice as much mass.
b) four times as much mass.
c) one-half the mass.
d) one-fourth the mass.
e) the same mass.

Explain.

Answer: e) The cart has the same mass. The periods of the systems in the two experiments are the same, so the masses have to be the same also.

(2) Compared to Experiment A, in Experiment C the cart has

a) twice as much mass.
b) four times as much mass.
c) one-half the mass.
d) one-fourth the mass.
e) the same mass.

Explain.

Answer: d) The cart has one-fourth the mass. The period of experiment C is half that of experiment A, so the mass is less for experiment C. Since the period is proportional to the square root of the mass, the mass in C has to be one-fourth that of A.

(3) Suppose that for a fourth experiment (Experiment D), the mass used in Experiment A was doubled and the spring was replaced with a spring with double the spring constant. The period in Experiment D would be

a) the same as the period in Experiment A.
b) double the period in Experiment A.
c) four times the period in Experiment A.
d) one-half the period in Experiment A.
e) one-fourth the period in Experiment A.

Explain.
Answer: a) The period would be the same in experiment D as it was in experiment A. Since the period is determined by the square root of the mass divided by the spring constant, doubling both will not change the period.
A cart attached to a spring is displaced from equilibrium and then released. A graph of velocity as a function of time for the cart is shown below. There is no friction.

(a) What is the period of the motion for this cart?

Explain.

Answer: The period is the time for the complete cycle of the oscillation so it is 8 seconds from the graph.

(b) What is the frequency of the motion for this cart?

Explain.

Answer: The period is 8 seconds from the graph. The frequency is the reciprocal of the period or 1/8 Hz.

(c) In which direction was the cart displaced from equilibrium before it was released?

Explain.

Answer: Since the velocity just after the cart was released is positive, the cart must be moving toward the right. It was displaced to the left.
A cart attached to a spring is displaced from equilibrium and then released. A graph of displacement as a function of time for the cart is shown below. There is no friction. Eight points are labeled A – H in the graph.

For each question below, choose from the labeled points above or state “none.”

1. At which point or points is the acceleration positive?
   Explain.
   Points D, E, and F. The acceleration is positive when the velocity is positive and increasing, or when the velocity is negative and decreasing. The velocity at point D is negative (the slope of the position graph is negative) and decreasing (the magnitude of the slope is smaller just after point D than it is just before point D); at point F the velocity is positive and increasing; and at point E the slope of the graph is instantaneously zero, but the slope is negative just before and positive just after, so the acceleration is positive.

2. At which point or points does the cart have zero velocity but nonzero net force?
   Explain.
   Points A and E. Since the velocity is zero at these points, but the velocity is changing, there is an acceleration, and therefore a net force.

3. At which point or points is the net force on the cart equal to zero?
   Explain.
   Points C and G. At these points the slope of the position graph is constant, so the velocity is constant. The acceleration is therefore zero, and the net force must be zero. From a dynamics perspective, at these points the cart is passing through the equilibrium position, and the spring exerts no force.

4. At which point or points are the acceleration, velocity, and displacement all positive?
   Explain.
   None. The acceleration is always negative when the position is positive. The force on the cart is always acting toward the equilibrium position; negative when the displacement is positive, and positive when the displacement is negative.

5. At which point or points is the acceleration nonzero and opposite in sign to the position?
   Explain.
   A, B, D, E, F, H. The acceleration is always opposite in sign to the position.

6. At which point or points is the velocity nonzero and opposite in sign to the acceleration?
   Explain.
   Points D and H. Whenever the cart is slowing down, the velocity and acceleration are in opposite directions.
A cart attached to a spring is displaced from equilibrium and then released. A graph of displacement as a function of time for the cart is shown. There is no friction. Points are labeled $A$ – $H$ in the graph.

For each question below, choose from the labeled points above or state “none” for the mass-spring-earth system.

1. At which point or points are the spring potential energy and the cart’s kinetic energy both at their maximum values?
   Explain. 
   *None. The spring potential energy and the cart’s kinetic energy always add up to the total energy, which is constant. So if one is at a maximum value, the other is at a minimum value.*

2. At which point or points is the kinetic energy equal to zero?
   Explain. 
   *Points $A$ and $E$. The slope of the graph is zero at these points indicating zero velocity.*

3. At which point or points is the total energy at its maximum value?
   Explain. 
   *$A$, $B$, $C$, $D$, $E$, $F$, $G$, and $H$. The total energy is always the same.*

4. At which point or points is the spring potential energy negative?
   Explain. 
   *None. The spring potential energy is proportional to the square of the distance the spring is stretched or compressed, and is always positive or zero.*

5. At which point or points is the kinetic energy positive?
   Explain. 
   *Points $B$, $C$, $D$, $F$, $G$ and $H$. The kinetic energy is always either positive or zero.*

6. At which point or points is the kinetic energy at its maximum value and the spring potential energy at its minimum value?
   Explain. 
   *Points $C$ and $G$. When the cart is in the equilibrium position the spring is unstretched and there is no spring potential energy. At these points the kinetic energy of the cart is equal to the total energy.*

7. At which point or points is the kinetic energy at its minimum value and the spring potential energy at its maximum value?
   Explain. 
   *Points $A$ and $E$. When the cart is at its maximum displacement the cart is momentarily at rest and the kinetic energy is zero.*
NT9D-CCT23: MASS OSCILLATING ON A VERTICAL SPRING—ENERGY

A mass hanging on a vertical spring is pulled down a distance $d$ and released. The mass undergoes simple harmonic motion. Three physics students make the following contentions about this situation:

Alexandra: “The maximum kinetic energy of this mass-spring system is fixed by the properties of the system and does not depend on how far down the mass is pulled. How far the mass is pulled will only affect the frequency of the oscillations.”

Bao: “No, that can’t be right since increasing the amplitude, or how far down it is pulled, increases the potential energy of the system. And I don’t think the amplitude has any effect on the frequency.”

Chung: “I agree in part with both of you. I think the amplitude does affect the maximum kinetic energy, but I also think it affects the frequency of the oscillations.”

Which, if any, of these three students do you agree with and think is correct?

Alexandra _____ Bao _____ Chung _____ None of them______

Please explain your reasoning.

Answer: Bao is correct since the maximum spring potential energy is proportional to the amplitude, and the maximum spring potential energy is equal to the maximum kinetic energy. And for simple harmonic motion the amplitude and frequency are unrelated.
A cart attached to a spring is given an initial push, displacing it from its equilibrium position. A graph of displacement as a function of time for the cart is shown at right. The system has a total initial energy of 12 J and there is no friction. Five points are labeled A – E in the graph.

For each labeled point, complete the bar chart below for the kinetic energy and the potential energy for the cart-spring system.

Explain.

At point A the cart is instantaneously at rest, so all of the energy is spring potential energy. At point B, which is halfway between the equilibrium point and the point of maximum displacement the spring potential energy is one-quarter of its maximum, so the kinetic energy is three-fourths of the maximum. At point C the spring is neither compressed nor stretched so there is no spring potential energy and the kinetic energy is at a maximum. At point D the energy values are the same as at B and at point E all of the energy is again spring potential.